

Whitney equisingularity of families of surfaces in \mathbb{C}^3

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Abstract

In this work, we study families of singular surfaces in \mathbb{C}^3 parametrized by \mathcal{A} -finitely determined map germs. We consider the topological triviality and Whitney equisingularity of an unfolding F of a finitely determined map germ $f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^3, 0)$. We investigate the following conjecture: topological triviality implies Whitney equisingularity of the unfolding F ? We provide a complete answer to this conjecture, given counterexamples showing how the conjecture can be false.

1 Introduction

Let $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^p, 0)$ be a finitely determined map germ. Given a 1-parameter unfolding $F : (\mathbb{C}^n \times \mathbb{C}, 0) \rightarrow (\mathbb{C}^p \times \mathbb{C}, 0)$ of f defined by $F(x, t) = (f_t(x), t)$, we assume it is origin preserving, that is, $f_t(0) = 0$ for any t . Then, we have a 1-parameter family of finitely determined map germs $f_t : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^p, 0)$.

By Thom's second isotopy lemma for complex analytic maps ([5], Theorem 5.2), every unfolding F of f which is Whitney equisingular is also topologically trivial. However, we know that the converse is not true in general (for instance, see [3], Example 6.2).

The work of Lê and Teissier at the beginning of the 1980s led to characterizations of Whitney equisingularity by the constancy of a finite sequence of polar multiplicities, (see [18] and [19]). Some time later, Gaffney [4] gave necessary and sufficient conditions to characterize the Whitney equisingularity of families of finitely determined map germs $f_t : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^p, 0)$. For families $f_t : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^3, 0)$, he showed the following result:

Theorem 1.1 ([4] Theorem 8.7 and Corollary 8.9) *If $f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^3, 0)$ is finitely determined and F is a 1-parameter unfolding of f , then :*

- (a) *F is Whitney equisingular if and only if $\mu(D(f_t))$, $m_1(f_t(\mathbb{C}^2), 0)$ and $m_0(f_t(D(f_t)))$ are constant.*
- (b) *If f has corank 1, F is Whitney equisingular if and only if $\mu(D(f_t))$ and $m_0(f_t(D(f_t)))$ are constant.*

where $D(f_t)$ is the double point curve of f_t , $m_1(f_t(\mathbb{C}^2), 0)$ is the first polar multiplicity of $f_t(\mathbb{C}^2)$ and $m_0(f_t(D(f_t)))$ is the Hilbert-Samuel multiplicity $e(m_{f_t(D(f_t))}, \mathcal{O}_{f_t(D(f_t))})$ where $m_{f_t(D(f_t))}$ is the maximal ideal of the local ring $\mathcal{O}_{f_t(D(f_t))}$ of $f_t(D(f_t))$.

We have already mentioned above that if F is Whitney equisingular then F is topologically trivial. The first author conjectured in 1994 (see [17]) that in the case of families of finitely determined map germs $f_t : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^3, 0)$ the converse is true. Moreover, conjectured that the unique necessary (and sufficient) invariant to control Whitney equisingularity is the Milnor number $\mu(D(f_t))$ of the double point curve of f_t .

Conjecture 1.2 ([17], Ruas's conjecture) *Let $f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^3, 0)$ be a finitely determined map germ and $F : (\mathbb{C}^2 \times \mathbb{C}, 0) \rightarrow (\mathbb{C}^3 \times \mathbb{C}, 0)$ a 1-parameter unfolding of f . Suppose that $\mu(D(f_t))$ is constant, then F is Whitney equisingular.*

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Unsuccessful attempts to prove this conjecture were presented in [1] and [16]. This problem remained unsolved for almost 25 years. In this work, we provide a complete answer to this question, given counterexamples which show different ways in which the conjecture may fail.

2 Double point spaces

Consider a finite and holomorphic map $f : U \rightarrow \mathbb{C}^3$, where U is a small enough neighbourhood of 0 in \mathbb{C}^2 . Throughout the paper, (x, y) and (X, Y, Z) are used to denote systems of coordinates in \mathbb{C}^2 and \mathbb{C}^3 , respectively. The double point curve of f , denoted by $D(f)$, is defined as the following set

$$D(f) := \{ (x, y) \in U : f^{-1}(f(x, y)) \neq \{(x, y)\} \} \cup \Sigma(f),$$

where $\Sigma(f)$ is the singular set of f . If f is finite and generically 1 – 1, then $D(f)$ is a closed analytic set of dimension 1. We also consider the lifting of the double point curve $D^2(f) \subset U \times U$ given by the pairs $((x, y), (x', y'))$ such that either $f(x, y) = f(x', y')$ with $(x, y) \neq (x', y')$ or $(x, y) = (x', y')$ and $(x, y) \in \Sigma(f)$ (see [7] and [9]).

We need to choose convenient analytic structures for the double point curve $D(f)$ and the lifting of the double point curve $D^2(f)$. To do this, we follow the construction of [7] which is also valid for holomorphic maps from \mathbb{C}^n to \mathbb{C}^p , with $n < p$.

Let us denote the diagonals in $\mathbb{C}^2 \times \mathbb{C}^2$ and $\mathbb{C}^3 \times \mathbb{C}^3$ by Δ_2 and Δ_3 , respectively, and denote the sheaves of ideals defining them by \mathcal{I}_2 and \mathcal{I}_3 , respectively. Locally, $\mathcal{I}_2 = \langle x - x', y - y' \rangle$ and $\mathcal{I}_3 = \langle X - X', Y - Y', Z - Z' \rangle$. Since the pull-back $(f \times f)^*\mathcal{I}_3$ is contained in \mathcal{I}_2 and U is small enough, then there exist functions $\alpha_{ij} \in \mathcal{O}_U$ well defined in all $U \times U$, such that

$$f_i(x, y) - f_i(x', y') = \alpha_{i1}(x, y, x', y')(x - x') + \alpha_{i2}(x, y, x', y')(y - y'), \text{ for } i = 1, 2, 3.$$

If $f(x, y) = f(x', y')$ and $(x, y) \neq (x', y')$, then every 2×2 minor of the matrix $\alpha = (\alpha_{ij})$ must vanish at (x, y, x', y') . We denote by $\mathcal{R}(\alpha)$ the ideal in $\mathcal{O}_{\mathbb{C}^4}$ generated by the 2×2 minors of α . Then we define the *lifting of the double point curve* $D(f)$ (with an analytic structure) as the complex space

$$D^2(f) = V((f \times f)^*\mathcal{I}_3 + \mathcal{R}(\alpha)).$$

And we call *double point ideal* the ideal $\mathcal{I}^2(f) = \langle (f \times f)^*\mathcal{I}_3 + \mathcal{R}(\alpha) \rangle$. Although the ideal $\mathcal{R}(\alpha)$ depends on the choice of the coordinate functions of f , in [12] it is proved that $\mathcal{I}^2(f)$ does not, and so $D^2(f)$ is well defined. It is easy to see that the points in the underlying set of $D^2(f)$ are exactly the ones in $U \times U$ of type (x, y, x', y') with $(x, y) \neq (x', y')$, $f(x, y) = f(x', y')$ and of type (x, y, x, y) such that (x, y) is a singular point of f .

Let $f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^3, 0)$ be a finite map germ and take a representative of f defined on a small enough open neighborhood of the origin. Denote by I_3 and $R(\alpha)$ the stalks at 0 of \mathcal{I}_3 and $\mathcal{R}(\alpha)$. We define the *lifting of the double point space of the map germ* f as the complex space germ $D^2(f) = V((f \times f)^*I_3 + R(\alpha))$.

Remark 2.1 Following Mond and Pellikan [13], given a finite morphism of complex spaces $f : X \rightarrow Y$ then the push-forward $f_*\mathcal{O}_X$ is a coherent sheaf of \mathcal{O}_Y -modules and $\mathcal{F}_k(f_*\mathcal{O}_X)$ denotes the k th Fitting ideal sheaf. Notice that the support of $\mathcal{F}_0(f_*\mathcal{O}_X)$ is just the image $f(X)$. Analogously, if $f : (X, x) \rightarrow (Y, y)$ is a finite map germ then we denote by $F_k(f_*\mathcal{O}_X)$ the k th Fitting ideal of $\mathcal{O}_{X,x}$ as $\mathcal{O}_{Y,y}$ -module.

We now describe an appropriate analytic structure to $D(f)$ and one more space that is important to study the topology of $f(\mathbb{C}^2)$, namely, the double point curve in the target, denoted by $f(D(f))$.

Definition 2.2 (a) Consider $f : U \rightarrow \mathbb{C}^3$ as above, and let $p : D^2(f) \subset U \times U \rightarrow U$ be the restriction to $D^2(f)$ of the projection of $U \times U$ on the first factor. The double point curve (with an analytic structure) is the complex curve

$$D(f) = V(\mathcal{F}_0(p_*\mathcal{O}_{D^2(f)})).$$

Set theoretically we have the equality $D(f) = p(D^2(f))$.

(b) The double point space in the target is the complex curve $f(D(f)) = V(\mathcal{F}_1(f_*\mathcal{O}_2))$. Notice that, the underlying set germ of $f(D(f))$ is the image of $D(f)$ by f .

(c) Given a finite map germ $f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^3, 0)$, the germ of the double point curve (with an analytic structure) is the germ of complex curve $D(f) = V(\mathcal{F}_0(p_*\mathcal{O}_{D^2(f)}))$. The germ of the double point curve in the image is the germ of the complex curve $f(D(f)) = V(\mathcal{F}_1(f_*\mathcal{O}_2))$, (for details, see [9] and [13]).

In [7], Marar and Mond showed that if $f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^3, 0)$ has corank 1, then f is finitely determined if and only if $\mu(D(f))$ is finite. The Milnor number of $D(f)$ is called *Mond number*. In [9], Marar, Nuño-Ballesteros and Peñafort-Sanchis extended this criterion of finite determinacy for the corank 2 case.

Theorem 2.3 ([9] Theorem 2.4 and Corollary 3.5) *Let $f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^3, 0)$ be a finite and generically 1 – 1 map germ. Then $D(f)$ is reduced if and only if $D^2(f)$ is reduced and the projection $p : D^2(f) \rightarrow (\mathbb{C}^2, 0)$ is generically 1 – 1. Hence, f is finitely determined if and only if its Mond number $\mu(D(f))$ is finite.*

Definition 2.4 *Let $f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^3, 0)$ be a finitely determined map germ. Take a representative $f : U \rightarrow V$ of f , where U and V are neighbourhoods of 0 in \mathbb{C}^2 and \mathbb{C}^3 and consider an irreducible component $D(f)^j$ of $D(f)$.*

(a) *If the restriction $f|_{D(f)^j} : D(f)^j \rightarrow \mathbb{C}^3$ is generically 1 – 1, we say that $D(f)^j$ is an identification component of $D(f)$. In this case, there exists an irreducible component $D(f)^i$ of $D(f)$, with $i \neq j$, such that $f(D(f)^j) = f(D(f)^i)$. We say that $D(f)^i$ is the correspondent identification component to $D(f)^j$ or that the pair $(D(f)^j, D(f)^i)$ is a pair of identification components of $D(f)$.*

(b) *If the restriction $f|_{D(f)^j} : D(f)^j \rightarrow \mathbb{C}^3$ is generically 2 – 1, we say that $D(f)^j$ is a fold component of $D(f)$.*

Example 2.5 *Let $f(x, y) = (x, y^2, xy^3 - x^3y)$ be the singularity C_3 of Mond's list ([12]). In this case, $D(f) = V(xy^2 - x^3)$. Then $D(f)$ has three irreducible components given by*

$$D(f)^1 = V(x), \quad D(f)^2 = V(x + y) \quad \text{and} \quad D(f)^3 = V(x - y).$$

Notice that $D(f)^1$ is a fold component, while $(D(f)^2, D(f)^3)$ is a pair of identification components. Also, we have that $f(D(f)^1) = V(Z, X)$ and $f(D(f)^2) = f(D(f)^3) = V(Z, Y - X^2)$ (see Figure 1).

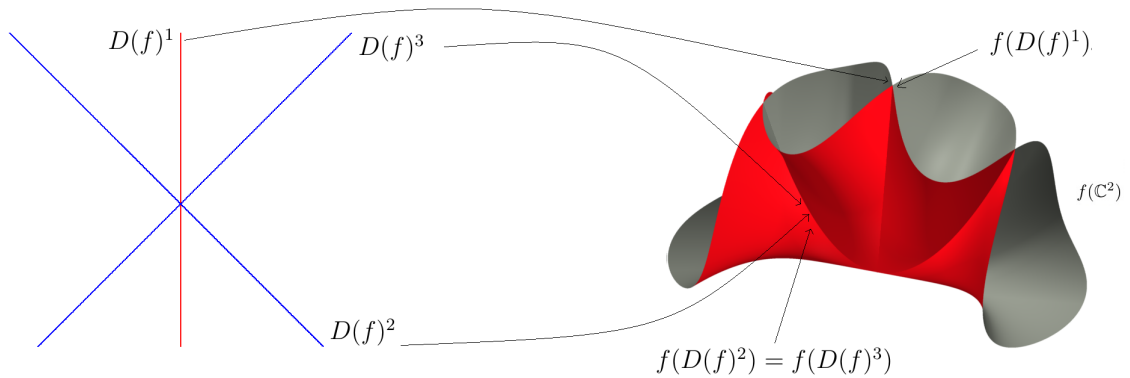


Figure 1: Identification and fold components of $D(f)$ (real points)

Let $f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^3, 0)$ be a corank 1 homogeneous map germ and write f in the form $f(x, y) = (x, p(x, y), q(x, y))$. We will see in Section 6 that if the degrees of p and q are even, then $D(f)$ is defined by a homogeneous function of odd degree, then $D(f)$ has exactly one fold component.

We will see in the next section that the Milnor number of $D(f_t)$ is an invariant that controls the topological triviality of the family of surfaces $f_t(\mathbb{C}^2)$.

3 Topological triviality of families of map germs

Following [1], we describe the necessary and sufficient conditions for a family f_t to be topologically trivial. Firstly, we will define two more invariants.

Let $f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^3, 0)$ be a finitely determined map germ and $F : (\mathbb{C}^2 \times \mathbb{C}, 0) \rightarrow (\mathbb{C}^3 \times \mathbb{C}, 0)$ a 1-parameter unfolding of f defined by $F(x, t) = (f_t(x), t)$. We assume that the origin is preserved, that is, $f_t(0) = 0$ for all t .

We say that a 1-parameter unfolding F is a stabilization of f if there is a representative $F : U \times T \rightarrow \mathbb{C}^3 \times T$, where T and U are open neighborhoods of 0 in \mathbb{C} and \mathbb{C}^2 respectively, such that $f_t : U \rightarrow \mathbb{C}^3$ is stable for all $t \in T \setminus \{0\}$.

By Mather-Gaffney's criterion [20], there is a representative $f : U \subset \mathbb{C}^2 \rightarrow V \subset \mathbb{C}^3$ such that $f^{-1}(0) = \{0\}$ and f is stable on $U \setminus \{0\}$, where V is an open neighborhoods of 0 in \mathbb{C}^3 . By shrinking U if necessary, we can assume that there are no cross-caps or triple points in $U \setminus \{0\}$. Then, since we are in the nice dimensions, we can take a stabilization of f , $F : U \times D \rightarrow \mathbb{C}^4$, $F(z, s) = (f_s(z), s)$ where D is a neighbourhood of 0 in \mathbb{C} . We are ready to give the next definition.

Definition 3.1 We define $C(f) = \sharp\{\text{cross-caps of } f_s\}$ and $T(f) = \sharp\{\text{triple points of } f_s\}$, for $s \neq 0$. These are analytic invariants of f and they can be computed as follows ([12]):

$$C(f) = \dim_{\mathbb{C}} \frac{\mathcal{O}_2}{Rf}, \quad T(f) = \dim_{\mathbb{C}} \frac{\mathcal{O}_3}{F_2(f_*\mathcal{O}_2)}$$

where Rf is the ideal generated by the maximal minors of the jacobian matrix of f .

Definition 3.2 Let $F : (\mathbb{C}^2 \times \mathbb{C}, 0) \rightarrow (\mathbb{C}^3 \times \mathbb{C}, 0)$ be a 1-parameter unfolding of a finitely determined map germ $f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^3, 0)$. We say that F is μ -constant if $\mu(D(f_t))$ is constant for all $t \in T$, where T is a neighborhood of 0 in \mathbb{C} .

Definition 3.3 Let $F : (\mathbb{C}^2 \times \mathbb{C}, 0) \rightarrow (\mathbb{C}^3 \times \mathbb{C}, 0)$ be a 1-parameter unfolding of a finitely determined map germ $f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^3, 0)$. We say that F is topologically trivial if there are germs of homeomorphisms:

$$\Phi : (\mathbb{C}^2 \times \mathbb{C}, 0) \rightarrow (\mathbb{C}^2 \times \mathbb{C}, 0), \quad \Phi(x, t) = (\phi_t(x), t), \quad \phi_0(x) = x, \quad \phi_t(0) = 0$$

$$\Psi : (\mathbb{C}^3 \times \mathbb{C}, 0) \rightarrow (\mathbb{C}^3 \times \mathbb{C}, 0) \quad \Psi(x, t) = (\psi_t(x), t), \quad \psi_0(x) = x, \quad \psi_t(0) = 0$$

such that $I = \Psi^{-1} \circ F \circ \Phi$, where $I(x, t) = (f(x), t)$ is the trivial unfolding of f .

The following theorem characterizes topological triviality.

Theorem 3.4 ([1] Theorem 6.2 and [2] Corollary 32) Let $F : (\mathbb{C}^2 \times \mathbb{C}, 0) \rightarrow (\mathbb{C}^3 \times \mathbb{C}, 0)$ be a 1-parameter unfolding of a finitely determined map germ $f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^3, 0)$. The following statements are equivalent:

- (1) F is topologically trivial.
- (2) F is μ -constant.

The following theorem has been shown in [7] in the case where f has corank 1 and in [9] in the case where f has corank 2. This result is very useful for the computations of the numerical invariants.

Theorem 3.5 ([7] Theorem 3.4 and [9] Theorem 4.3) *Let $f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^3, 0)$ be finitely determined. Then*

- (a) $\mu(D(f)) = \mu(D^2(f)) + 6T(f)$
- (b) $\mu(D^2(f)) = 2\mu(D^2(f)/S_2) + C(f) - 1$
- (c) $\mu(D(f)) = 2\mu(f(D(f))) + C(f) - 2T(f) - 1.$

where $D^2(f)/S_2$ is the quotient complex space of $D^2(f)$ by the group $S_2 = \mathbb{Z}/2$.

The next corollary follows immediately by the upper semi-continuity of the invariants $D^2(f)$, $D^2(f)/S_2$, $C(f)$ and $T(f)$ and the formulas in Theorem 3.5.

Corollary 3.6 *Let $F : (\mathbb{C}^2 \times \mathbb{C}, 0) \rightarrow (\mathbb{C}^3 \times \mathbb{C}, 0)$ be a 1-parameter unfolding of a finitely determined map germ $f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^3, 0)$. If $\mu(D(f_t))$ is constant, then $C(f_t)$ and $T(f_t)$ are constant.*

Theorem 3.7 ([14]) *Let $f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^3, 0)$ be a quasi-homogeneous finitely determined map germ. Write $f(x, y) = (f_1, f_2, f_3)$ and denote by d_i the degree of f_i . Let ω_1, ω_2 be the weights of x and y , respectively. If $\epsilon = d_1 + d_2 + d_3 - \omega_1 - \omega_2$ and $\delta = d_1 d_2 d_3 / (\omega_1 \omega_2)$, then*

$$C(f) = \frac{1}{\omega_1 \omega_2} ((d_2 - \omega_1)(d_3 - \omega_2) + (d_1 - \omega_2)(d_3 - \omega_2) + (d_1 - \omega_1)(d_2 - \omega_1)),$$

$$T(f) = \frac{1}{6\omega_1 \omega_2} (\delta - \epsilon)(\delta - 2\epsilon) + \frac{C(f)}{3}, \quad \mu(D(f)) = \frac{1}{\omega_1 \omega_2} (\delta - \epsilon - \omega_1)(\delta - \epsilon - \omega_2),$$

4 Whitney equisingularity of families of surfaces in \mathbb{C}^3

Gaffney defined in [4] the excellent unfoldings. An excellent unfolding has a natural stratification whose strata in the complement of the parameter space T are the stable types in source and target. For families F as in previous section, the strata in the source are the following

$$\{\mathbb{C}^2 \setminus D^2(F), D^2(F) \setminus T, T\}$$

In the target, the strata are:

$$\{\mathbb{C}^3 \setminus F(\mathbb{C}^2 \times \mathbb{C}), F(\mathbb{C}^2 \times \mathbb{C}) \setminus \overline{F(D^2(F))}, F(D^2(F)) \setminus T, T\}.$$

Notice that F preserves the stratification, that is, F sends a stratum into a stratum.

Definition 4.1 *An unfolding F as above is Whitney equisingular if the above stratifications in source and target are Whitney equisingular along T .*

In addition to Theorem 1.1, the following results are known:

Theorem 4.2 ([1] Theorem 8.7 and [4] Theorem 5.2) *Let F be a 1-parameter unfolding of a finitely determined map germ $f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^3, 0)$. If F is μ -constant, then F is an excellent unfolding.*

In [8], Marar and Nuño-Ballesteros show that in the corank 1 case the Whitney equisingularity of f_t is characterized by the constancy of three invariants $C(f_t)$, $J(f_t)$ and $T(f_t)$, where $J(f_t)$ is the number of tacnodes that appear in a generic perturbation of the transversal section of f_t .

Theorem 4.3 ([8] Corollary 4.7) *Let F be a 1-parameter unfolding of a finitely determined map germ $f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^3, 0)$ with corank 1. Then, the following conditions are equivalent:*

(a) F is Whitney equisingular.

(b) $C(f_t)$, $J(f_t)$ and $T(f_t)$ are constant.

Remark 4.4 Also in [8], they show that if $f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^3, 0)$ is finitely determined, then $C(f) + 2J(f) + 6T(f) = \mu(D(f)) + 2m_0(f(D(f))) - 1$. Therefore, the constancy of the invariants on the left side of the equality occurs if and only if $\mu(D(f))$ and $m_0(f(D(f)))$ are constant.

Another useful result is the following lemma:

Lemma 4.5 ([4], Proposition 8.4 and [9], Lemma 5.2) *Let $f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^3, 0)$ be a finitely determined map germ. If $H \subset \mathbb{C}^3$ is a generic plane, $Y_0 = H \cap f(\mathbb{C}^2)$ and \tilde{Y}_0 is the plane curve in $(\mathbb{C}^2, 0)$ given by $\tilde{Y}_0 = f^{-1}(H)$, we have:*

$$(a) \ m_1(f(\mathbb{C}^2), 0) = \mu(\tilde{Y}_0, 0) + m_0(f(\mathbb{C}^2), 0) - 1.$$

$$(b) \ \mu_1(f(\mathbb{C}^2), 0) = \mu(\tilde{Y}_0, 0) + 2 \cdot m_0(f(D(f)), 0).$$

Item (a) was shown in [4]. Item (b) appears in [9], in which it is shown that the Whitney equisingularity of f_t is characterized by the constancy of only two invariants, $\mu(D(f_t))$ and $\mu_1(f_t(\mathbb{C}^2), 0)$.

Theorem 4.6 ([9] Theorem 5.3) *Let F be a 1-parameter unfolding of a finitely determined map germ $f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^3, 0)$. Then, F is Whitney equisingular if and only if $\mu(D(f_t))$ and $\mu_1(f_t(\mathbb{C}^2), 0)$ are constant.*

The invariant $\mu_1(f(\mathbb{C}^2), 0)$ is defined in [9] as follows. As in Lemma 4.5, let H be a generic plane in \mathbb{C}^3 , $Y_0 := f(\mathbb{C}^2) \cap H$ and $\hat{Y}_0 := f^{-1}(H)$. For a generic H , $(Y_0, 0)$ and $(\hat{Y}_0, 0)$ are germs of reduced plane curves. Then, $\mu_1(f(\mathbb{C}^2), 0) := \mu(Y_0, 0)$. Let $F = (f_t(x), t)$ be an unfolding of f , for the next section, we will adopt the following notation, $(Y_t, 0) := (f_t(\mathbb{C}^2) \cap H, 0)$ and $(\hat{Y}_t, 0) := (f_t^{-1}(H), 0)$.

5 Counterexamples to Ruas's conjecture

In this section, we use Mond-Pellikaan's algorithm in [12] to find a presentation matrix of a finite analytic map germ $g : (X, 0) \rightarrow (\mathbb{C}^{n+1}, 0)$, where (X, x) is a germ of Cohen-Macaulay analytic space of dimension n . For the computations we have made use of the software Singular [21] and the implementation of Mond-Pellikaan's algorithm given by Hernandez, Miranda, and Peñafort-Sanchis in [6]. At the webpage of Miranda [10] one can find a Singular library to compute presentation matrices based on the results of [6].

The following example shows that in corank 2 case the conjecture is false.

Example 5.1 *Let $f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^3, 0)$ be defined by:*

$$f(x, y) = (x^2, x^2y + xy^2 + y^3, x^5 + y^5)$$

First we show that f is finitely determined. Denote by (x, y, x', y') a point in $\mathbb{C}^2 \times \mathbb{C}^2$ and by q^n the homogeneous function $q^n : \mathbb{C}^2 \rightarrow \mathbb{C}$ defined by $q^n(w, w') = w^n + w^{n-1}w' + \dots + ww'^{n-1} + w'^n$. Take a representative $f : U \rightarrow \mathbb{C}^3$ of the germ f defined in a small enough neighborhood U of 0. Then the double point ideal $\mathcal{I}^2(f)$ is generated by $h_1, \dots, h_6 \in \mathbb{C}\{x, y, x', y'\}$, where:

$$h_1(x, y, x', y') = (x + x')(x - x'),$$

$$h_2(x, y, x', y') = q^4(y, y')(x + x')$$

$$h_3(x, y, x', y') = q^4(x, x')(x - x') + q^4(y, y')(y - y'), \quad h_4(x, y, x', y') = (x + x') \left(2q^2(y, y') + (x + x')(y + y') + (x^2 + x'^2) \right)$$

$$h_5(x, y, x', y') = q^4(y, y') \left((y + y')(x + x') + (y^2 + y'^2) \right) - q^4(x, x') \left(2q^2(y, y') + (x + x')(y + y') + (x^2 + x'^2) \right)$$

$$h_6(x, y, x', y') = \left((y + y')(x + x') + (y^2 + y'^2) \right) (x - x') + \left(2q^2(y, y') + (x + x')(y + y') + (x^2 + x'^2) \right) (y - y')$$

The projection $p : D^2(f) \subset U \times U \rightarrow U$, is just $(x, y, x', y') \mapsto (x, y)$. Then the ideal $\mathcal{F}_0(p_*\mathcal{O}_{D^2(f)})$ is generated by the determinant of the following presentation matrix 22×22 of $p_*\mathcal{O}_{D^2(f)}$, obtained by means of the Singular library mentioned above:

$$\begin{pmatrix} y & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & y & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & x^2 & g_{14} & 0 & 0 & -x^2 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & y & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ x^3 & x^2 & x & x^2 & g_{14} & x^2 & x & 0 & x & 1 & x & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & y & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & y & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x^2 & g_{14} & 0 & 0 & -x^2 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & y & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x^3 & x^2 & x & x^2 & g_{14} & x^2 & x & 0 & x & 1 & x & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & y & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & y & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x^2 & g_{14} & 0 & 0 & -x^2 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & y & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & g_{36} & 0 & g_1 & 0 & g_{38} & g_2 & 0 & xy & 0 & x^2 & x & x^2 & g_{15} & x^2 & x & 0 & -x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & y & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & g_{42} & -x^2y & 0 & 0 & g_{39} & 0 & g_3 & -x^3 & x^2 & -x & -x^2 & 0 & 0 & g_{16} & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & g_{37} & 0 & -g_1 & 0 & -x^3 & -x^2 & 0 & -xy & 0 & 0 & 0 & 0 & -x & 0 & 0 & y & x & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & y & -1 & 0 & 0 & 0 \\ 0 & 0 & g_5 & 0 & g_6 & 0 & -x^4 & g_7 & g_{42} & g_8 & g_{21} & g_{23} & g_{24} & g_{25} & g_{26} & 0 & g_{27} & 0 & g_{28} & g_{17} & g_{29} & g_{20} \\ 0 & 0 & g_{41} & 0 & g_{40} & 0 & -x^4 & g_9 & 0 & -g_1 & x^4 & -x^3 & g_{19} & 0 & 0 & 0 & 0 & 0 & x^2 & x & g_{14} & x \\ 0 & 0 & g_{11} & 0 & g_{12} & 0 & x^4 & g_{13} & 2x^4 & g_{14} & g_{22} & g_{35} & g_{15} & g_{30} & g_{31} & 0 & g_{32} & 0 & g_{33} & g_{34} & g_{29} & g_{18} \end{pmatrix}$$

where

$$\begin{aligned} g_1 &= x^2y + xy^2, \quad g_2 = 2x^2 + xy, \quad g_3 = x^2 + xy, \quad g_4 = \frac{7}{10}x^3y + \frac{11}{10}xy^3, \quad g_5 = \frac{-9}{10}x^2y^2 - \frac{9}{5}xy^3, \\ g_6 &= \frac{-13}{10}x^3 - \frac{7}{10}x^2y + \frac{2}{5}xy^2, \quad g_7 = \frac{-11}{10}x^2y - \frac{3}{2}xy^2, \quad g_8 = -2x^3 - 2x^2y, \quad g_9 = \frac{-3}{10}x^3y - \frac{9}{10}xy^3, \\ g_{10} &= \frac{11}{10}x^2y^2 + \frac{11}{10}xy^3, \quad g_{11} = \frac{7}{10}x^3 - \frac{7}{10}x^2y - \frac{8}{5}xy^2, \quad g_{12} = \frac{9}{10}x^2y + \frac{3}{2}xy^2, \quad g_{13} = \frac{17}{5}x^2 - xy, \quad g_{14} = x + y, \\ g_{15} &= 2x + y, \quad g_{16} = -x + y, \quad g_{17} = \frac{-7}{5}x + y, \quad g_{18} = \frac{17}{10}x + y, \quad g_{19} = -x^2 - xy, \quad g_{20} = \frac{-13}{10}x, \quad g_{21} = \frac{11}{10}x^4, \\ g_{22} &= \frac{-9}{10}x^4, \quad g_{23} = \frac{7}{5}x^3, \quad g_{24} = \frac{13}{5}x^2, \quad g_{25} = \frac{19}{10}x^3, \quad g_{26} = \frac{6}{5}x^2, \quad g_{27} = \frac{-14}{5}xy, \quad g_{28} = \frac{13}{10}x^2, \quad g_{29} = \frac{x}{10}, \\ g_{30} &= \frac{-21}{10}x^3, \quad g_{31} = \frac{4}{5}x^2, \quad g_{32} = \frac{11}{5}xy, \quad g_{33} = \frac{-17}{10}x^2, \quad g_{34} = \frac{8}{5}x, \quad g_{35} = \frac{8}{5}x^3, \quad g_{36} = -xy^2, \quad g_{37} = xy^2, \quad g_{38} = 2x^3, \\ g_{39} &= 2x^2, \quad g_{40} = -x^2y^2, \quad g_{41} = -x^3y \text{ and } g_{42} = -2x^4. \end{aligned}$$

Then,

$$\begin{aligned} D(f) &= V(\mathcal{F}_0(p_*\mathcal{O}_{D^2(f)})) = V(3x^{22} + 4x^{21}y + 8x^{20}y^2 + 22x^{19}y^3 + 57x^{18}y^4 + 120x^{17}y^5 + 154x^{16}y^6 + 82x^{15}y^7 - \\ &141x^{14}y^8 - 484x^{13}y^9 - 790x^{12}y^{10} - 988x^{11}y^{11} - 1064x^{10}y^{12} - 1098x^9y^{13} - 1102x^8y^{14} - 1026x^7y^{15} - 837x^6y^{16} - \\ &578x^5y^{17} - 333x^4y^{18} - 160x^3y^{19} - 65x^2y^{20} - 20xy^{21} - 5y^{22}). \end{aligned}$$

Notice that $\mathcal{F}_0(p_*\mathcal{O}_{D^2(f)})$ is a homogeneous function of degree 22. Moreover, it is not difficult to check with the help of a computer that it has isolated singularity. This implies that f is finitely determined by Theorem 2.3. Now, consider the following 1-parameter unfolding $F = (f_t(x, y), t)$ of f defined by:

$$f_t(x, y) = (x^2 + txy, x^2y + xy^2 + y^3, x^5 + y^5)$$

We have that f is homogeneous and that the unfolding F only adds terms of same degree. This implies that F is topologically trivial by [3]. Alternatively, a calculation shows that $\mu(D(f_t)) = 441$ for all t , then by Theorem 3.4 F is topologically trivial. Choose constants a, b and c in \mathbb{C} with $a \neq 0$ such that the plane H defined by $H = V(aX + bY + cZ)$ is generic. In this way the family $(Y_t, 0)$ is given by

$$(Y_t, 0) = V(a(x^2 + txy) + b(x^2y + xy^2 + y^3) + c(x^5 + y^5))$$

Then, $\mu(Y_0, 0) = 2$ and $\mu(Y_t, 0) = 1$ for $t \neq 0$. By Lemma 4.5 and the upper semicontinuity of the invariants, $\mu_1(f_t(\mathbb{C}^2))$ can not be constant. Hence F is not Whitney equisingular by Theorem 4.6.

Using the formulas in Definition 3.1 and Theorem 3.5, we have that $C(f_t) = 14$, $T(f_t) = 56$ and $\mu(f_t(D(f_t))) = 270$ for all t .

The multiplicities $m_0(f_t(D(f_t)))$ and $m_0(f_t(\mathbb{C}^2))$ remain constant. In fact, $m_0(f_t(D(f_t))) = 22$ and $m_0(f_t(\mathbb{C}^2)) = 6$ for all t . Moreover, by Lemma 4.5 it follows that $\mu_1(f(\mathbb{C}^2), 0) = 46$, $\mu_1(f_t(\mathbb{C}^2), 0) = 45$ for $t \neq 0$, $m_1(f(\mathbb{C}^2), 0) = 7$ and $m_1(f_t(\mathbb{C}^2), 0) = 6$ for $t \neq 0$.

Remark 5.2 In [15], Peñafort-Sanchis shows that if $n, m, k \geq 2$ are coprime integers, then the map germ

$$f(x, y) = (x^n, y^m, (x + y)^k)$$

is finitely determined. So a way to find a class of topologically trivial unfoldings with the property that $\mu(\hat{Y}_t, 0)$ is not constant, as in the previous example, is the following:

Let $n, m, k \geq 2$ be coprime integers, with $n < m < k$, and consider the map germ $f(x, y) = (x^n, y^m, (x + y)^k)$. Let be $F = (f_t(x, y), t)$ defined by

$$f(x, y) = (x^n + ty^n, y^m, (x + y)^k)$$

The map germ f is homogeneous and the unfolding F only adds terms of same degree. Again, this implies that F is topologically trivial by [3]. Also,

$$\mu(\tilde{Y}_0, 0) = (n - 1)(m - 1) > (n - 1)(n - 1) = \mu(\tilde{Y}_t, 0), \text{ for } t \neq 0,$$

then F is not Whitney equisingular by Theorem 4.6.

The following two examples show that if an unfolding F of f is topologically trivial, the multiplicity $m_0(f_t(D(f_t)))$ does not have to be constant. In the first f has corank 2, and in the second f has corank 1.

Example 5.3 Consider the map germ $f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^3, 0)$ defined by

$$f(x, y) = (x^3, y^5, x^2 - xy + y^2)$$

It is not difficult to compute the double point curve $D(f)$ with the help of a computer and verify that f is finitely determined, by just following the same routine calculations used in Example 5.1. In fact, $D(f)$ is a reduced curve given by

$$D(f) = V((81x^{16} - 324x^{15}y + 945x^{14}y^2 - 1971x^{13}y^3 + 3384x^{12}y^4 - 4716x^{11}y^5 + 5625x^{10}y^6 - 5664x^9y^7 + 5026x^8y^8 - 3900x^7y^9 + 2810x^6y^{10} - 1840x^5y^{11} + 1155x^4y^{12} - 625x^3y^{13} + 300x^2y^{14} - 100xy^{15} + 25y^{16})(x^4 - 3x^3y + 4x^2y^2 - 2xy^3 + y^4)(x^2 - xy + y^2))$$

Now, consider the following unfolding $F = (f_t(x, y), t)$ of f

$$f_t(x, y) := (x^3, y^5, x^2 - xy + y^2 + tx^2)$$

Notice that f is homogeneous and the unfolding F only adds terms of same degree, then F is topologically trivial. The presentation matrix of the push forward $f_{t*}(\mathcal{O}_2)$ is given by

$$\begin{pmatrix} -Z & 0 & 1+t & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ gX & -Z & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & gX & -Z & X & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -Z & 0 & 1+t & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & gX & -Z & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & gX & -Z & X & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -Z & 0 & 1+t & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & gX & -Z & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & gX & -Z & X & 0 & 0 & 0 & 0 & 1 \\ Y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -Z & 0 & 1+t & 0 & -1 & 0 \\ 0 & Y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & gX & -Z & 0 & 0 & 0 & -1 \\ 0 & 0 & Y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & gX & -Z & X & 0 & 0 \\ 0 & Y & 0 & Y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -Z & 0 & 1+t \\ 0 & 0 & Y & 0 & Y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & gX & -Z & 0 \\ -XY & 0 & 0 & 0 & 0 & Y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & gX & -Z \end{pmatrix}$$

where $g = (1+t)$, then

$$f_t(\mathbb{C}^2) = V(F_0(f_{t*}(\mathcal{O}_2))) = V(Y^6 + h_1XY^5Z + h_2X^5Y^3 + h_3X^2Y^4Z^2 + h_4X^6Y^2Z + h_5X^3Y^3Z^3 - 3Y^4Z^5 + h_6X^{10} + h_7X^7YZ^2 + h_8X^4Y^2Z^4 + h_9XY^3Z^6 + h_{10}X^8Z^3 + h_{11}X^5YZ^5 + h_{12}X^2Y^2Z^7 + h_{13}X^6Z^6 + h_{14}X^3YZ^8 + 3Y^2Z^{10} + h_{15}X^4Z^9 + h_{16}XYZ^{11} + h_{17}X^2Z^{12} - Z^{15})$$

where

$$\begin{aligned} h_1(x, y) &= 30t + 15, & h_2(x, y) &= 15t^7 - 35t^6 - 147t^5 - 135t^4 - 20t^3 + 30t^2 + 15t + 2, & h_3(x, y) &= 30t^3 + 315t^2 + 315t + 90, \\ h_4(x, y) &= 15t^9 - 90t^8 - 735t^7 - 1785t^6 - 2070t^5 - 1125t^4 - 75t^3 + 225t^2 + 105t + 15, \\ h_5(x, y) &= 375t^4 + 1600t^3 + 2145t^2 + 1215t + 245, \\ h_6(x, y) &= t^{15} + 15t^{14} + 105t^{13} + 455t^{12} + 1365t^{11} + 3003t^{10} + 5005t^9 + 6435t^8 + 6435t^7 + 5005t^6 + 3003t^5 + 1365t^4 + 455t^3 \\ &\quad + 105t^2 + 15t + 1, \\ h_7(x, y) &= -90t^{10} - 750t^9 - 2745t^8 - 5760t^7 - 7560t^6 - 6300t^5 - 3150t^4 - 720t^3 + 90t^2 + 90t + 15, \\ h_8(x, y) &= 135t^6 + 1260t^5 + 3900t^4 + 5775t^3 + 4500t^2 + 1800t + 300, & h_9(x, y) &= 15t - 80, \\ h_{10}(x, y) &= -5t^{12} - 60t^{11} - 330t^{10} - 1100t^9 - 2475t^8 - 3960t^7 - 4620t^6 - 3960t^5 - 2475t^4 - 1100t^3 - 330t^2 - 60t - 5, \\ h_{11}(x, y) &= 135t^7 + 1035t^6 + 3222t^5 + 5310t^4 + 4995t^3 + 2655t^2 + 720t + 72, & h_{12}(x, y) &= 90t^3 + 45t^2 - 330t - 345, \\ h_{13}(x, y) &= 10t^9 + 90t^8 + 360t^7 + 840t^6 + 1260t^5 + 1260t^4 + 840t^3 + 360t^2 + 90t + 10, \\ h_{14}(x, y) &= -225t^3 - 720t^2 - 765t - 270, & h_{15}(x, y) &= -10t^6 - 60t^5 - 150t^4 - 200t^3 - 150t^2 - 60t - 10, \\ h_{16}(x, y) &= -45t - 60, & h_{17}(x, y) &= 5t^3 + 15t^2 + 15t + 5. \end{aligned}$$

Consider the plane H defined by $X - Z = 0$ which in this example is generic to f . Then, the family of plane curves $(f_t(\mathbb{C}^2) \cap H) = (Y_t, 0)$ is defined in the coordinates U and W of $(\mathbb{C}^2, 0) \simeq H$ by

$$(Y_t, 0) = V(W^6 + h_1U^2W^5 + h_2U^5W^3 + h_3U^4W^4 + h_4U^7W^2 + h_5U^6W^3 - 3U^5W^4 + h_6U^{10} + h_7U^9W + h_8U^8W^2 + h_9U^7W^3 + h_{10}U^{11} + h_{11}U^{10}W + h_{12}U^9W^2 + h_{13}U^{12} + h_{14}U^{11}W + 3U^{10}W^2 + h_{15}U^{13} + h_{16}U^{12}W + h_{17}U^{14} - U^{15})$$

Then, $\mu(Y_0, 0) = 47$ and $\mu(Y_t, 0) = 45$ for $t \neq 0$, hence F is not Whitney equisingular by Theorem 4.6. Notice that $\mu(\tilde{Y}_t, 0) = 1$ for all t . Moreover, by Lemma 4.5, we have that $m_0(f(D(f))) = 23$ and $m_0(f_t(D(f_t))) = 22$ for $t \neq 0$.

Notice also that the pair of irreducible components $D(f)^1 := V(x - \alpha y)$ and $D(f)^2 := V(x - \beta y)$, where α and β are complex numbers such that $(x - \alpha y)(x - \beta y) = x^2 - xy + y^2$, is a pair of identification components of $D(f)$.

One of the main ideas of this example is that the map germ f was constructed in such way that the image of the identification components $D(f)^1$ and $D(f)^2$ have multiplicity 3, while all the other irreducible components of $f(D(f))$ have multiplicity 2. When we take an appropriate deformation f_t of f , all the irreducible components of $f_t(D(f_t))$ now have multiplicity 2, for $t \neq 0$.

We now present the following counterexample to the conjecture in the corank 1 case.

Example 5.4 Let f be the map germ defined by $f(x, y) = (x, y^4, x^5y - 5x^3y^3 + 4xy^5 + y^6)$. As in the previous examples, it is not difficult to check, with the help of a computer, that f is finitely determined. Now, consider the following unfolding $F = (f_t(x, y), t)$ of f

$$f_t(x, y) = (x, y^4, x^5y - 5x^3y^3 + 4xy^5 + y^6 + ty^7)$$

The map germ f is homogeneous and its unfolding F only adds terms of same degree, then F is topologically trivial. The presentation matrix of the push forward $f_{t*}(\mathcal{O}_2)$ is given by

$$\begin{pmatrix} -Z & X^5 + 4XY & Y & -5X^3 + tY \\ -5X^3Y + tY^2 & -Z & X^5 + 4XY & Y \\ Y^2 & -5X^3Y + tY^2 & -Z & X^5 + 4XY \\ X^5Y + 4XY^2 & Y^2 & -5X^3Y + tY^2 & -Z \end{pmatrix}$$

Then

$$\begin{aligned} f_t(D(f_t)) = & V(YZ^2 - Y^4 + 16X^2Y^2Z + 8tXY^4 + t^2Y^3Z - 40X^4Y^3 - 10tX^3Y^2Z + 33X^6YZ + 2tX^5Y^3 - 10X^8Y^2 + X^{10}Z, \\ & Z^3 - Y^3Z - 16X^2Y^4 - 8tXY^3Z - t^2Y^5 + 40X^4Y^2Z + 10tX^3Y^4 - 33X^6Y^3 - 2tX^5Y^2Z + 10X^8YZ - X^{10}Y^2, \\ & 8XY^2Z + tYZ^2 + tY^4 - 5X^3Z^2 + 59X^3Y^3 - 4t^2XY^4 + 2X^5YZ + 40tX^4Y^3 - 52X^7Y^2 - t2X^5Y^3 + 10tX^8Y^2 - 13X^{11}Y + X^{15}, \\ & 8XY^4 + 2tY^3Z - 74X^3Y^2Z - 48tX^2Y^4 - 4t^2XY^3Z + X^5Z^2 + 241X^5Y^3 + t^3Y^5 + 40tX^4Y^2Z - 15t^2X^3Y^4 - \\ & 132X^7YZ + 59tX^6Y^3 - 45X^9Y^2 - 4X^{11}Z - tX^{10}Y^2 + 5X^{13}Y) \end{aligned}$$

We can check that $m_0(f(D(f))) = 9$ and $m_0(f_t(D(f_t))) = 8$ for $t \neq 0$. Then F is not Whitney equisingular by Theorem 4.6. Also, as $\mu(D(f_t)) = 196$, $C(f_t) = 15$ and $T(f_t) = 20$ for all t , by Remark 4.4 it follows that $J(f) = 39$ and $J(f_t) = 38$ for $t \neq 0$. Then the constancy of the invariant $\mu(D(f_t))$ does not imply the constancy of the invariant $J(f_t)$.

In this example, the curve $D(f)^1$ defined by $x = 0$ is a fold component of $D(f)$. The image by f of the component $D(f)^1$ has multiplicity 2, while all the other irreducible components of $f(D(f))$ have multiplicity 1. See Figure 2 that shows five real irreducible components of $D(f)$ denoted by $D(f)^2 = V(x - y)$, $D(f)^3 = V(x + y)$, $D(f)^4 = V(x + 2y)$, $D(f)^5 = V(x - 2y)$ (the others are not real components and do not appear in the figure). We create the appropriate deformation f_t of f so that all the irreducible components of $f_t(D(f_t))$ have multiplicity 1, for $t \neq 0$.

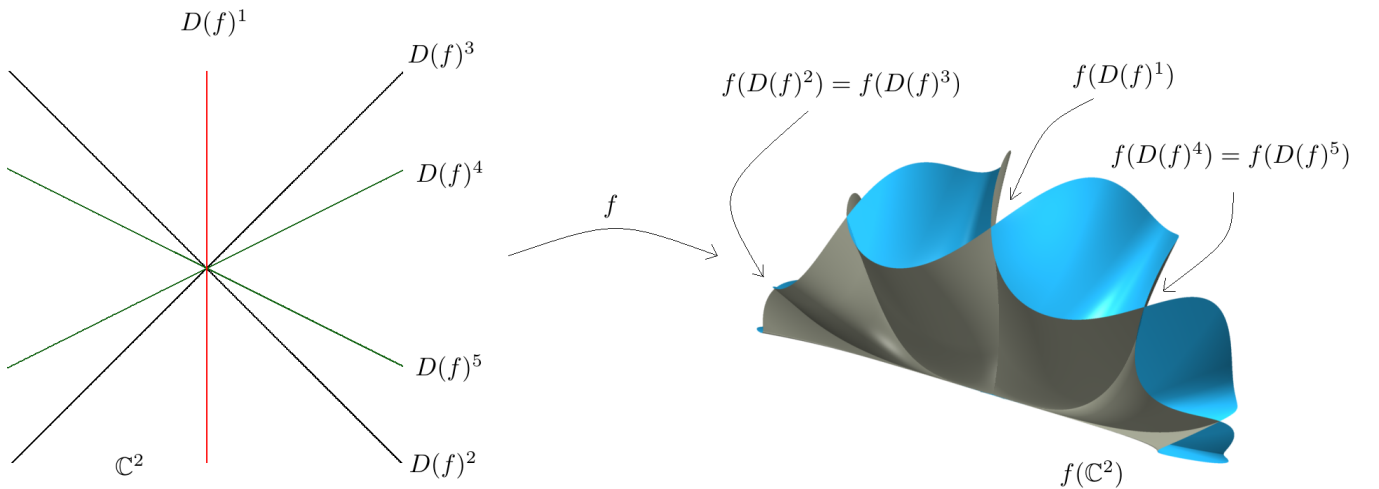


Figure 2: Five real components of $D(f)$ and the real image of $f(\mathbb{C}^2)$.

We conclude this section with Table 1 which shows other counterexamples for the conjecture. The reader can check the details of each example following the same routine performed in the examples described above.

Family of map germs	$\mu(\tilde{Y}_0, 0)$	$\mu(\tilde{Y}_t, 0)$	$m_0(f(D(f)))$	$m_0(f_t(D(f_t)))$
<hr/> Corank 1 <hr/>				
$(x, y^4, x^5y + xy^5 + y^6 + ty^7)$	0	0	9	8
$(x, y^4, x^9y + xy^9 + y^{10} + ty^{11})$	0	0	15	14
$(x, y^4, x^{13}y + xy^{13} + y^{14} + ty^{15})$	0	0	21	20
$(x, y^4, x^{17}y + xy^{17} + y^{18} + ty^{19})$	0	0	27	26
$(x, y^6, x^7y + xy^7 + y^8 + ty^9)$	0	0	20	18
$(x, y^6, x^{13}y + xy^{13} + y^{14} + ty^{15})$	0	0	35	33
<hr/> Corank 2 <hr/>				
$(x^2 + txy, x^2y + xy^2 + y^3, x^5 + y^5)$	2	1	22	22
$(x^3, y^5, x^2 - xy + y^2 + tx^2)$	1	1	23	22
$(x^n + ty^n, y^m, (x + y)^k)$ (with $2 \leq n < m < k$ and n, m, k coprimes in pairs) and $d = nmk - n - m - k + 2$	$(n - 1)(m - 1)$	$(n - 1)^2$	$\frac{dn}{2}$	$\frac{dn}{2}$

Table 1: Counterexamples to the conjecture 1.2

6 Some formulas

All examples of the previous section are deformations of homogeneous map germs. Inspired by them, we present in this section some formulas for the invariants $m_0(f(D(f)))$, $\mu_1(f(\mathbb{C}^2))$ and $J(f)$ in the case which f is finitely determined, homogeneous and has corank 1. We need the following lemma.

Lemma 6.1 *Let $\alpha : \mathbb{C} \rightarrow \mathbb{C}^n$ be a map defined by*

$$\alpha(t) = (a_1 t^{m_1}, a_2 t^{m_2}, \dots, a_n t^{m_n}),$$

Then α is generically d -to-1 over the image $\alpha(\mathbb{C})$, where $d = \gcd(m_1, m_2, \dots, m_n)$, the greatest common divisor of m_1, m_2, \dots, m_n .

Proof. It is not difficult to see that if $\gcd(m_1, m_2, \dots, m_n) = 1$ then α is generically 1-to-1 over its image. When $\gcd(m_1, m_2, \dots, m_n) \neq 1$, write $m_1 = dq_1, \dots, m_n = dq_n$ and consider the maps $\beta : \mathbb{C} \rightarrow \mathbb{C}$ and $\gamma : \mathbb{C} \rightarrow \mathbb{C}^n$ defined by

$$\beta(t) = t^d \quad \text{and} \quad \gamma(z) = (a_1 z^{q_1}, \dots, a_n z^{q_n}).$$

Then $\alpha = \gamma \circ \beta$, β is generically d -to-1, hence α is generically d -to-1. ■

Proposition 6.2 *Let $f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^3, 0)$ be a homogeneous finitely determined map germ of corank 1. Write f in the form $f(x, y) = (x, p(x, y), q(x, y))$ and denote by n and m the degrees of p and q , respectively, with $2 \leq n \leq m$.*

(a) *$D(f)$ is the germ of a homogeneous curve with d smooth irreducible components, where $d = nm - n - m + 1$ and*

$$D(f) = V\left(\prod_{i=1}^d (x - \alpha_i y)\right)$$

where $\alpha_i \in \mathbb{C}$.

(b) d is odd if and only if n and m are both even.

(c) If n and m are both even, then $\gcd(n, m) = 2$, otherwise f is not finitely determined. In this case, $D(f)$ has $d - 1$ identification components, and $D(f)^j = V(x)$ is the unique fold component of $D(f)$ and the following hold:

$$\begin{aligned} m_0(f(D(f))) &= \frac{nm - m}{2} & \mu_1(f(\mathbb{C}^2)) &= nm - m \\ J(f) &= \frac{m^2n + mn^2 - m^2 - 7mn - n^2 + 6m + 7n - 6}{2} \end{aligned}$$

(d) If n and m are not both even, then each of the d components of $D(f)$ is an identification component. In this case, we have that

$$\begin{aligned} m_0(f(D(f))) &= \frac{nm - n - m + 1}{2} & \mu_1(f(\mathbb{C}^2)) &= nm - n - m + 1 \\ J(f) &= \frac{m^2n + mn^2 - m^2 - 7mn - n^2 + 6m + 6n - 5}{2} \end{aligned}$$

Proof. (a) The proof that $D(f)$ is homogeneous and $d = nm - n - m + 1$ follows from ([14], Proposition 1.15). We have that $D(f) = V(\lambda(x, y))$, where $\lambda(x, y)$ is a homogeneous polynomial of degree d . In this way, we can write

$$\lambda(x, y) = \prod_{i=1}^d (\beta_i x - \gamma_i y), \quad \text{with} \quad \beta_i, \gamma_i \in \mathbb{C}$$

Let $f : U \rightarrow \mathbb{C}^3$ be a representative of f and suppose that $D(f)^1 = V(y)$ is an irreducible component of $D(f)$. Consider the map $n_1 : U_1 \rightarrow U$, defined by $n_j(t) = (t, 0)$, where U_1 is a small enough open neighborhood of 0 in \mathbb{C} . The map $f \circ n_1$ is generically 1-to-1, then $D(f)^1$ is an identification component of $D(f)$. Any other component $D(f)^i$ of $D(f)$ is given by $D(f)^i = V(x - \alpha_i y)$, where $\alpha_i = \gamma_i / \beta_i$, then $f(D(f)^1) \cap f(D(f)^i) = \{0\}$, a contradiction. Then we can rewrite λ in the form

$$\lambda = \prod_{i=1}^d (x - \alpha_i y),$$

since $\beta_i \neq 0$, for all i .

(b) Write $n = 2a$ and $m = 2b$, then $nm - n - m + 1 = (2a)(2b) - (2a) - (2b) + 1$ which is odd. Suppose now that $n = 2a$ and $m = 2b + 1$, then $(2a)(2b + 1) - (2a) - (2b + 1) + 1 = 4ab - 2b$ which is even. Finally, suppose that $n = 2a + 1$ and $m = 2b + 1$, then $(2a + 1)(2b + 1) - (2a + 1) - (2b + 1) + 1 = 4ab - 2$ which is even.

(c) Consider the parametrization $n_j : U_j \rightarrow \mathbb{C}^2$ of each irreducible component $D(f)^j = V(x - \alpha_j y)$ of $D(f)$, defined by $n_j(t) = (\alpha_j t, t)$. If $\alpha_j \neq 0$, $f \circ n_j$ is generically 1-to-1, then $D(f)^j$ is an identification component of $D(f)$. Since n and m are both even, d is odd, then $D(f)$ has exactly $d - 1$ identification components and 1 fold component, denoted by $D(f)^1$, which is necessarily given by $D(f)^1 = V(x)$. Write f in the form

$$f(x, y) = (x, a_0 x^n + a_1 x^{n-1} y + \cdots + a_n y^n, b_0 x^m + b_1 x^{m-1} y + \cdots + b_m y^m)$$

Notice that $a_n \neq 0$ or $b_m \neq 0$, otherwise f is not finite. We have three cases:

- (c.1) $f \circ n_1(t) = (a_n t^n, b_m t^m)$, if $a_n \neq 0$ and $b_m \neq 0$
- (c.2) $f \circ n_1(t) = (a_n t^n, 0)$, if $a_n \neq 0$ and $b_m = 0$
- (c.3) $f \circ n_1(t) = (0, b_m t^m)$, if $a_n = 0$ and $b_m \neq 0$

Let's look at each case.

Case (c.1): Let be $c = \gcd(n, m)$, the map $f \circ n_1$ is generically c -to-1, hence $c = 2$, otherwise f is not finitely determined. Since $n \leq m$, it follows that $m_0(f(D(f)^1))) = \frac{n}{2}$.

Case (c.2): We have that $f \circ n_1(t) = (a_n t^n, 0)$ is generically 2-to-1, hence $n = 2$. We can make coordinate changes at the source and target (see [11]) such that f can be written in the form

$$f(x, y) = (x, y^2, y h(x, y^2))$$

where $h(x, y^2)$ is a homogeneous polynomial of degree $m - 1$. Since $2 = n \leq m$, it follows that $m_0(f(D(f)^1))) = \frac{n}{2}$.

Case (c.3): This case is analogous to the previous case.

In any case we have that $\gcd(n, m) = 2$ and $m_0(f(D(f)^1))) = \frac{n}{2}$. Then

$$m_0(f(D(f))) = \left(\frac{d-1}{2} \right) + \frac{n}{2} = \frac{nm-n}{2}$$

Using the formulas in Lemma 4.5, Remark 4.4 and Theorem 3.7, we obtain the corresponding formulas for $\mu_1(f(\mathbb{C}^2))$ and $J(f)$.

(d) Consider again the parametrization $n_j : U_j \rightarrow \mathbb{C}^2$ of each component $D(f)^j = V(x - \alpha_j y)$ of $D(f)$, defined by $n_j(t) = (\alpha_j t, t)$. If $\alpha_j \neq 0$, $f \circ n_j$ is generically 1-to-1, hence $D(f)^j$ is an identification component of $D(f)$. If there exists some fold component of $D(f)$, the unique possibility is when $\alpha_j = 0$, then $D(f)$ can have only one fold component or each component of $D(f)$ is an identification component. As n and m are not both even, d must be even. Since the number of identification components of $D(f)$ is always even, the number of fold components of $D(f)$ is even, so each component of $D(f)$ is an identification component. Since the image of each identification component is a smooth curve in \mathbb{C}^3 , we have that

$$m_0(f(D(f))) = \frac{nm - n - m + 1}{2} = \frac{d}{2}$$

Again, using the formulas in Lemma 4.5, Remark 4.4 and Theorem 3.7, we obtain the corresponding formulas for $\mu_1(f(\mathbb{C}^2))$ and $J(f)$. ■

Proposition 6.3 *Let $f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^3, 0)$ be the map germ defined by $f(x, y) = (x^n, y^m, (x+y)^k)$, where n, m and k are coprime in pairs. If $n < m < k$, then $D(f)$ has $d = nmk - n - m - k + 2$ smooth irreducible components and each component of $D(f)$ is an identification component. Furthermore*

$$m_0(f(D(f))) = \frac{dn}{2} \quad \text{and} \quad \mu_1(f(\mathbb{C}^2)) = (n-1)(m-1) + dn$$

Proof. By ([14], Proposition 1.15) it follows that $D(f)$ is a homogeneous curve with $d = nmk - n - m - k + 2$ smooth irreducible components. Since n, m and k are coprime in pairs and $n < m < k$, d is always even. Let $D(f)^j = V(\beta_j x - \gamma_j y)$ be an irreducible component of $D(f)$ and consider its parametrization $n_j : U_j \rightarrow \mathbb{C}^2$. Again, since n, m, k are coprime in pairs, the map $f \circ n_j$ is generically 1-to-1, then $D(f)^j$ is an identification component of $D(f)$. Notice that $V(x)$ and $V(y)$ can not be irreducible components of $D(f)$. So the image of each curve $D(f)^j$ by f is a curve in \mathbb{C}^3 with multiplicity n , and

$$m_0(f(D(f))) = \frac{dn}{2}$$

Since $\mu(\tilde{Y}_0, 0) = (n-1)(m-1)$, using Lemma 4.5, we obtain that $\mu_1(f(\mathbb{C}^2)) = (n-1)(m-1) + dn$. ■

7 A case in which the conjecture is true

We now present a class of families of map germs in which the conjecture is true. Before, we will need the following lemma.

Lemma 7.1 *Let $f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^3, 0)$ be a finitely determined map germ of corank 1. Let $F = (f_t, t)$ be a topologically trivial 1-parameter unfolding of f . If every irreducible component of $f(D(f))$ is smooth, then F is Whitney equisingular.*

Proof. Since the multiplicity $m_0(f(D(f)^j))$ of each irreducible component of $f(D(f))$ is 1, the result follows by the upper semi-continuity of the multiplicity. ■

Theorem 7.2 *Let $f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^3, 0)$ be a homogeneous finitely determined map germ of corank 1. Write f in the form $f(x, y) = (x, p(x, y), q(x, y))$ and let n and m be the degrees of p and q , respectively. Let $F = (f_t, t)$ be a topologically trivial 1-parameter unfolding of f . If $\gcd(n, m) \neq 2$, then F is Whitney equisingular.*

Proof. By Proposition 6.2, it follows that all components of $D(f)$ are identification components. Then the image of each component $D(f)^j$ by f is a smooth curve in \mathbb{C}^3 , and the result follows by Lemma 7.1. ■

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